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Maximal graded orders over crystalline graded rings

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ABSTRACT

Crystalline graded rings are generalizations of certain classes of rings like generalized twisted group rings, generalized Weyl algebras, and generalized skew crossed products. Under certain conditions, in particular, the group is finite, it is proven that the global dimension of a crystalline graded ring coincides with the global dimension of its base ring. When, in addition, the base ring is a commutative Dedekind domain, two constructions are given for producing maximal graded orders. On the way, a new concept appears, so-called, spectrally twisted group. Some general properties of it are studied. At the end of the paper several examples are considered.

Introduction

Crystalline graded rings were introduced in [9] as a generalization of generalized twisted group rings as well as of generalized Weyl algebras studied in [1–4]. In this paper we focus attention on crystalline graded rings, graded by a finite group, where the base ring (part of degree 0) is a commutative Dedekind domain.

An obvious problem is to determine whether such a crystalline graded ring is a maximal graded order, knowing from the situation in case of generalized twisted group rings that they need not be maximal orders. In Section 2 we prove that the global dimension of a crystalline graded ring, in case the order of the grading group is invertible in the ring equals the global dimension of the base ring, leading in some cases to a proof of the semiprimeness of the ring.

As a tool for the section on maximal orders, we introduce a non-associative generalization of a group, the so-called spectrally twisted group in Section 3.

Section 4 deals with the maximal graded order structure of a crystalline graded ring. Using the prime decomposition of the ideals generated by the 2-cocycle describing the crystalline grading, each

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prime of the commutative Dedekind domain corresponds to a 2-cocycle in $H^2(G,\mathbb{Z})$ in case the primes are invariant under the action on the part of degree 0. In case the action is not trivial on the primes a new "kind" of 2-cocycle appears, i.e. a spectrally twisted 2-cocycle. We generalize the property that the 2-cocycle in $H^2(G,\mathbb{Z})$ is equivalent to a 2-cocycle taking values in $\{0,1\}$, to the twisted situation, cf. [6,8].

Such twisted 2-cocycles lead to the construction of maternal orders, but these need not be maximal graded orders now. We describe an iterative method of constructing orders over a maternal order. We also consider the special case where the primes in the decomposition of the ideals generated by the 2-cocycle values, $D\alpha(\sigma,\tau)$ for $\sigma,\tau \in G$, the grading group, are invariant for the action of G on D (the degree zero part). Then, the maternal order is the unique maximal graded order containing the crystalline graded ring, cf. [6]. We provide some examples, a first one of a maternal order that is maximal, a second one where the maternal order is contained in two different maximal graded orders, and a third one where 3 different maternal orders appear, two of them being maximal graded orders and allowing a conjugation that does not contain the crystalline graded ring.

1. Preliminaries on crystalline graded rings

Throughout, we will work with associative rings A with unit element 1_A, or if there is no ambiguity, we will just write 1.

Definition 1 *(Pre-crystalline graded ring).* Let G be an arbitrary group. Let $u: G \to A$ be a map of sets (injective) such that $u_e = 1$, with e the neutral element of G, and $u_g \neq 0$ for all $g \in G$. Let A_0 be an associative ring with $1_{A_0} = 1_A$. We have the following properties:

1. $A = \bigoplus_{g\in G} A_0 u_g$.
2. For every $g \in G$, $A_0 u_g = u_g A_0$ and this is a free left A_0-module of rank 1.
3. The decomposition $A = \bigoplus_{g\in G} A_0 u_g$ makes A into a G-graded ring with $A_0 = A_e$.

We call an A which satisfies all these conditions a **pre-crystalline graded ring**.

This definition implies, $\forall a_0 \in A_0$:

1. From Definition 1(2) we find there exists a set map σ defined by

$$\sigma : G \to \mathrm{End}(A_0) : g \mapsto \sigma_g,$$

   where $\forall g \in G$, σ_g is defined by

$$u_g a_0 = \sigma_g(a_0) u_g.$$

   As it turns out, $\forall g \in G$, σ_g is an epimorphism. See [7].
2. Since A is associative, we find a map $\alpha : G \times G \to A_0$. It satisfies $\forall g,h,t \in G$:

$$\alpha(g,h)\alpha(gh,t) = \sigma_g\big(\alpha(h,t)\big)\alpha(g,ht).$$

   We call this the cocycle relation.

Given the set of symbols $\{u_g \mid g \in G\}$, the maps σ and α as defined above, we can construct a crystalline ring given a base ring A_0 as follows:

$$A_0 \mathop{\diamond}_{\sigma,\alpha} G := \bigoplus_{g\in G} A_0 u_g.$$

Definition 2. Notation as above, define the set $S(G)$ by

$$S(G) = \{\alpha(g, g^{-1}) \mid g \in G\}.$$

For a subset X of A, define the set $t_{S(G)}(X)$ by

$$t_{S(G)}(X) = \{x \in X \mid \exists s \in S(G)\colon\ sx = 0\}.$$

A subset X of A is called $S(G)$-torsionfree if $t_{S(G)}(X) = \{0\}$.

We have the following propositions (see [7]):

Proposition 3. *Notation as above, the following are equivalent*:

1. *A_0 is $S(G)$-torsionfree.*
2. *A is $S(G)$-torsionfree.*
3. *$\alpha(g, g^{-1})a_0 = 0$ for some $g \in G$ implies $a_0 = 0$.*
4. *$\alpha(g, h)a_0 = 0$ for some $g, h \in G$ implies $a_0 = 0$.*
5. *$A_0 u_g = u_g A_0$ is also free as a right A_0-module with basis u_g for every $g \in G$.*
6. *For every $g \in G$, σ_g is bijective hence a ring automorphism of A_0.*

Proposition 4. *If A_0 is $S(G)$-torsionfree then the following statements are equivalent*:

1. *For $g, h \in G$, $\alpha(g^{-1}, g) \in Z(A_0)$, resp. $\alpha(g, h) \in Z(A_0)$.*
2. *For $g, h \in G$, $\sigma_{g^{-1}}\sigma_g = \mathrm{Id}$, resp. $\sigma_g\sigma_h = \sigma_{gh}$.*

Definition 5. A pre-crystalline graded ring A that is $S(G)$-torsionfree is called a crystalline graded ring (CGR). Additionally, in case $\alpha(g, h) \in Z(A_0), \forall g, h \in G$, we say that A is centrally crystalline graded.

Let A be crystalline graded. Let P be a prime ideal in A_0, $x \in G$. Since the map σ_x as defined above is a homomorphism, the set $\sigma_x^{-1}(P)$ is also a prime ideal, implying an action of G on $\mathrm{Spec}(A_0)$. To ease notation in the rest of the article, we write for $x \in G$, $P \in \mathrm{Spec}(A_0)$:

$$\sigma_x^{-1}(P) = Px.$$

In other words, G acts on the right of $\mathrm{Spec}(A_0)$.

Lemma 6. *Let $A_0 \underset{\sigma,\alpha}{\diamond} G$ be a pre-crystalline graded ring, $x \in A_0$, $g, h \in G$. Then*:

1. $\sigma_g^{-1}(x)u_g^{-1} = u_g^{-1}x$.
2. $\sigma_{hg}^{-1}[\alpha(h, g)] = \sigma_g^{-1}[\sigma_h^{-1}(\alpha(h, g))]$.
3. $\sigma_g^{-1}[\alpha(g, g^{-1}h)] = \alpha^{-1}(g^{-1}, h)\sigma_g^{-1}[\alpha(g, g^{-1})]$.

Proof.

1.
$$\begin{aligned}
&\sigma_g[\sigma_h(x)]\alpha(g, h) = \alpha(g, h)\sigma_{gh}(x)\\
&\quad\Rightarrow\quad \sigma_g[\sigma_{g^{-1}}(x)]\alpha(g, g^{-1}) = \alpha(g, g^{-1})x\\
&\quad\Rightarrow\quad \sigma_{g^{-1}}(x)\sigma_g^{-1}(\alpha(g, g^{-1})) = \sigma_{g^{-1}}(\alpha(g, g^{-1}))\sigma_g^{-1}(x)\\
&\quad\Rightarrow\quad \sigma_g^{-1}(x) = \sigma_g^{-1}[\alpha^{-1}(g, g^{-1})]\sigma_{g^{-1}}(x)\sigma_g^{-1}[\alpha(g, g^{-1})],
\end{aligned}$$

so

$$\begin{aligned}\sigma_g^{-1}(x)u_g^{-1} &= \sigma_g^{-1}\big[\alpha^{-1}\big(g,g^{-1}\big)\big]\sigma_{g^{-1}}(x)\sigma_g^{-1}\big[\alpha\big(g,g^{-1}\big)\big]\alpha^{-1}\big(g^{-1},g\big)u_{g^{-1}}\\ &= \sigma_g^{-1}\big[\alpha^{-1}\big(g,g^{-1}\big)\big]\sigma_{g^{-1}}(x)\alpha\big(g^{-1},g\big)\alpha^{-1}\big(g^{-1},g\big)u_{g^{-1}}\\ &= \sigma_g^{-1}\big[\alpha^{-1}\big(g,g^{-1}\big)\big]\sigma_{g^{-1}}(x)u_{g^{-1}}\\ &= \sigma_g^{-1}\big[\alpha^{-1}\big(g,g^{-1}\big)\big]u_{g^{-1}}x\\ &= \alpha^{-1}\big(g,g^{-1}\big)u_{g^{-1}}x\\ &= u_g^{-1}x.\end{aligned}$$

2.
$$\begin{aligned}&\sigma_h\big[\sigma_g(x)\big]\alpha(h,g) = \alpha(h,g)\sigma_{hg}(x)\\ \Rightarrow\quad &\sigma_h\big[\sigma_g\big(\sigma_{hg}^{-1}\big(\alpha(h,g)\big)\big)\big]\alpha(h,g) = \alpha(h,g)\sigma_{hg}\big(\sigma_{hg}^{-1}\big(\alpha(h,g)\big)\big)\\ \Rightarrow\quad &\sigma_{hg}^{-1}\big[\alpha(h,g)\big] = \sigma_g^{-1}\big[\sigma_h^{-1}\big(\alpha(h,g)\big)\big].\end{aligned}$$

3.
$$\alpha\big(g,g^{-1}\big)\alpha(e,h) = \sigma_g\big[\alpha\big(g^{-1},h\big)\big]\alpha\big(g,g^{-1}h\big). \qquad \square$$

Proposition 7. *Let $A = R \mathop{\diamond}\limits_{\sigma,\alpha} G$ be a CGR,*

$$H = \big\{g \in G \;\big|\; \alpha\big(g,g^{-1}\big) \in U(D)\big\}.$$

Then the following properties hold:

1. *For $h \in H$ and $x \in G$, $\alpha(x,h)$ and $\alpha(h,x)$ are invertible in R.*
2. *H is a subgroup of G.*
3. *For $h, h' \in H$ and $x, y \in G$*:

$$R\alpha\big(hx, yh'\big) = R\sigma_h\big(\alpha(x,y)\big).$$

In particular, if $\alpha(x,y)$ is invertible in R then so is $\alpha(hx, yh')$ for every $h, h' \in H$.

Proof. 1. For $h \in H$ and $x \in G$ we have the 2-cocycle relation:

$$\alpha(x,h)\alpha\big(xh,h^{-1}\big) = \sigma_x\big(\alpha\big(h,h^{-1}\big)\big)\alpha(x,e).$$

The right-hand side is invertible in R, hence $\alpha(x,h)$ is invertible too because the right inverse yields a left inverse ($\alpha(x,h)$ is normalizing in R) and obviously these two will then have to coincide. On the other hand, the 2-cocycle relation for (h^{-1},h,x) yields

$$\alpha\big(h^{-1},h\big)\alpha(e,x) = \sigma_{h^{-1}}\big(\alpha(h,x)\big)\alpha\big(h^{-1},hx\big),$$

where $\alpha(h^{-1},h)$ is invertible in R because $h \in H$, and $\alpha(h^{-1},hx)$ is invertible in R by the foregoing since $h^{-1} \in H$. Consequently $\sigma_{h^{-1}}(\alpha(h,x))$ and therefore also $\alpha(h,x)$ is invertible in R.

2. Consider $x, y \in H$ and look at the 2-cocycle relation for $(x, y, y^{-1}x^{-1})$, i.e.:

$$\alpha(x,y)\alpha\big(xy,y^{-1}x^{-1}\big) = \sigma_x\big(\alpha\big(y,y^{-1}x^{-1}\big)\big)\alpha\big(x,x^{-1}\big),$$

where $\alpha(x, x^{-1})$ and $\sigma_x(\alpha(y, y^{-1}x^{-1}))$ are invertible in R because $x, y \in H$ (using 1). It follows that $\alpha(xy, y^{-1}x^{-1})$ is invertible in R too.

3. From 1 it follows that $Ru_{hx} = Ru_h Ru_x$, for $h \in H$. Then:

$$Ru_{hx}u_{yh'} = R\alpha\big(hx, yh'\big)u_{hxyh'}.$$

But on the other hand

$$\begin{aligned}
Ru_{hx}u_{yh'} &= Ru_h u_x Ru_y u_{h'} \\
&= Ru_h u_x u_y u_{h'} \\
&= Ru_h \alpha(x, y) u_{xy} u_{h'} \\
&= R\sigma_h\big(\alpha(x, y)\big)u_h u_{xy} u_{h'} \\
&= R\sigma_h\big(\alpha(x, y)\big)u_h u_{xyh'} = R\sigma_h\big(\alpha(x, y)\big)u_{hxyh'}
\end{aligned}$$

applying the remark starting the proof of 3. Since $Ru_{hxyh'}$ is free (on the left) it follows that $R\alpha(hx, yh') = R\sigma_h(\alpha(x, y))$. Finally $\alpha(x, y)$ is invertible in R if and only if $\sigma_h(\alpha(x, y))$ is invertible if and only if $R = R\alpha(hx, yh')$, i.e. when $\alpha(hx, yh')$ is invertible. □

For more properties on crystalline graded structures, see [7].

2. Global dimension of crystalline graded structures

Theorem 8. *Let R, S be rings with $R \subseteq S$ such that R is an R-bimodule direct summand of S, then* $\operatorname{r\,gld} R \leqslant \operatorname{r\,gld} S + \operatorname{pd} S_R$.

Proof. See [5, p. 237]. □

Theorem 9. *Let A_0 be a ring, G a finite group with $|G|$ a unit in A_0 and $A = A_0 \underset{\sigma,\alpha}{\diamond} G$ a pre-crystalline graded ring with u_g units. (A is almost crossed product.) Let M be any right A-module. Then*:

1. *If $K \lhd M_A$ and K is a direct summand of M as an A_0-module, then K is a direct summand over A.*
2. $\operatorname{pd} M_{A_0} = \operatorname{pd} M_A$.
3. $\operatorname{r\,gld} A_0 = \operatorname{r\,gld} A$.

Proof. 1. Let $\pi : M \to K$ be the A_0-module splitting morphism. Define $\lambda : M \to K$ by

$$\lambda(m) = |G|^{-1} \sum_{g \in G} \pi(mu_g)u_g^{-1}.$$

λ **is well defined**: trivial.
λ **is the identity on** K:

$$\begin{aligned}
\lambda(k) &= |G|^{-1} \sum_{g \in G} \pi(ku_g)u_g^{-1} \\
&= |G|^{-1} \sum_{g \in G} k = k.
\end{aligned}$$

λ **is** A**-linear**: Let $m \in M$, $a \in A$

$$\begin{aligned}
\lambda(ma) &= |G|^{-1} \sum_{g \in G} \pi(mau_g)u_g^{-1} \\
&= |G|^{-1} \sum_{g \in G} \pi\left[m\left(\sum_{h \in G} t_h u_h\right)u_g\right]u_g^{-1} \\
&= |G|^{-1} \sum_{g,h \in G} \pi(mt_h u_h u_g)u_g^{-1} \\
\text{(Lemma 6(1))} &= |G|^{-1} \sum_{g,h \in G} \pi(mu_h u_g)u_g^{-1}\sigma_h^{-1}(t_h) \\
&= |G|^{-1} \sum_{g,h \in G} \pi\big(m\alpha(h,g)u_{hg}\big)u_g^{-1}\sigma_h^{-1}(t_h) \\
&= |G|^{-1} \sum_{g,h \in G} \pi(mu_{hg})\sigma_{hg}^{-1}\big[\alpha(h,g)\big]u_g^{-1}\sigma_h^{-1}(t_h) \\
\text{(Lemma 6(2))} &= |G|^{-1} \sum_{g,h \in G} \pi(mu_{hg})\sigma_g^{-1}\big[\sigma_h^{-1}\big(\alpha(h,g)\big)\big]u_g^{-1}\sigma_h^{-1}(t_h) \\
\text{(Lemma 6(1))} &= |G|^{-1} \sum_{g,h \in G} \pi(mu_{hg})u_g^{-1}\sigma_h^{-1}\big[\alpha(h,g)\big]\sigma_h^{-1}(t_h) \\
(x=hg) &= |G|^{-1} \sum_{h \in G}\sum_{x \in G} \pi(mu_x)u_{h^{-1}x}^{-1}\sigma_h^{-1}\big[\alpha\big(h,h^{-1}x\big)\big]\sigma_h^{-1}(t_h) \\
\text{(Lemma 6(3))} &= |G|^{-1} \sum_{h \in G}\sum_{x \in G} \pi(mu_x)\big[\alpha^{-1}\big(h^{-1},x\big)u_{h^{-1}}u_x\big]^{-1} \\
&\quad \cdot \alpha^{-1}\big(h^{-1},x\big)\sigma_h^{-1}\big[\alpha\big(h,h^{-1}\big)\big]\sigma_h^{-1}(t_h) \\
&= |G|^{-1} \sum_{h \in G}\sum_{x \in G} \pi(mu_x)u_x^{-1}u_{h^{-1}}^{-1}\sigma_h^{-1}\big[\alpha\big(h,h^{-1}\big)\big]\sigma_h^{-1}(t_h) \\
&= |G|^{-1} \sum_{h \in G}\sum_{x \in G} \pi(mu_x)u_x^{-1}u_h\sigma_h^{-1}(t_h) \\
&= |G|^{-1} \sum_{x \in G} \pi(mu_x)u_x^{-1} \sum_{h \in G} t_h u_h \\
&= \lambda(m) \cdot a.
\end{aligned}$$

2. Suppose M_{A_0} is projective and

$$0 \to K \to F \to M \to 0$$

is a short exact sequence of A-modules with F free. Then the sequence splits over A_0 and hence over A by 1. So M_A is also projective. Furthermore, A_{A_0} is free. Now, an A-projective resolution of any module M_A is also an A_0-projective resolution that terminates when a kernel is, equally, A_0-projective or A-projective, so $\operatorname{pd} M_{A_0} = \operatorname{pd} M_A$.

3. Any A-module is naturally an A_0-module. Since $\operatorname{pd} M_{A_0} = \operatorname{pd} M_A$:

$$\begin{aligned} \text{r gld}\, A &= \sup\{\operatorname{pd} M_A \mid M_A \text{ right } A\text{-module}\} \\ &\leqslant \sup\{\operatorname{pd} M_{A_0} \mid M_{A_0} \text{ right } A_0\text{-module}\} \\ &= \text{r gld}\, A_0. \end{aligned}$$

By Theorem 8:

$$\begin{aligned} \text{r gld}\, A_0 &\leqslant \text{r gld}\, A + \operatorname{pd} A_{A_0} \\ &\stackrel{2}{=} \text{r gld}\, A + \operatorname{pd} A_A \\ &= \text{r gld}\, A. \end{aligned}$$

In conclusion: r gld $A_0 =$ r gld A. □

Definition 10. A multiplicative subset S of a ring R is said to be right Ore if $\forall s \in S$, $a \in R$:

1. $aS \cap sR \neq \emptyset$.
2. If $sa = 0$, then $\exists u \in S$ such that $au = 0$.

If S is a right Ore set, one can construct the ring of right fractions RS^{-1} similar as in the commutative case. A left Ore set is defined analogously.

Lemma 11. *Let $A = A_0 \underset{\sigma,\alpha}{\diamond} G$ be a CGR, where G is a finite group. Then the set of regular elements in A_0,* reg A_0, *is a subset of* reg A, *the regular elements of A.* (*The regular elements are the elements which are not zero dividers.*) *Furthermore, if A_0 is a semiprime Goldie Ring,* reg A_0 *is left* (*and right*) *Ore in A. We have*

$$(\operatorname{reg} A_0)^{-1} A = \bigoplus_{g \in G} \mathbb{Q}_{\text{cl}}(A_0) u_g$$

where $\mathbb{Q}_{\text{cl}}(A_0)$ is the classical quotient field of A_0.

Proof. First we prove $\operatorname{reg} A_0 \subset \operatorname{reg} A$. Take $a \in \operatorname{reg} A_0$, $x = \sum_{g \in G} x_g u_g$ and suppose $ax = 0$. Then $\sum_{g \in G} a x_g u_g = 0$. This implies $a x_g = 0$, $\forall g \in G$, and then x_g, $\forall g \in G$. Suppose $xa = 0$, then $\sum_{g \in G} x_g u_g a = 0$. This implies $x_g \sigma_g(a) u_g = 0$, or $x_g \sigma_g(a) = 0$, $\forall g \in G$. $\operatorname{reg} A_0$ is invariant under σ_g, $\forall g \in G$, therefore $x_g = 0$, $\forall g \in G$. In other words, $\operatorname{reg} A_0 \subset \operatorname{reg} A$.

For the second part, we prove $\operatorname{reg} A_0$ is left and right Ore in A. By Goldie's Theorem, $\operatorname{reg} A_0$ is an Ore set in A_0. First the left Ore condition for A is proven for $S = \operatorname{reg} A_0$. Concretely, we need to show $\forall r \in R$, $s \in S$, $\exists r' \in R$, $s' \in S$ such that $s'r = r's$. Let $r = \sum_{g \in G} a_g u_g$. Since S is left Ore for R, we can find for all $g \in G$ elements $a'_g \in R$ and $s_g \in S$ such that $a'_g \sigma_g(s) = s_g a_g$. It is well known that $\exists s' \in S \cap \bigcap_{g \in G} R s_g$. In other words, \$$\exists s' \in S$ and $\exists v_g \in R$ such that for all $g \in G$, $s' = v_g s_g$. Set $\forall g \in G$, $b_g = v_g a'_g$ and set $r' = \sum_{g \in G} b_g u_g$. Then $r's = s'r$. The right Ore condition is similar. The third assertion is now clear. □

Theorem 12. *Let A be a CGR over A_0, A_0 a semiprime Goldie Ring, G a finite group. Assume* char A_0 *does not divide $|G|$, then A is a semiprime Goldie Ring.*

Proof. Since A is a CGR, the elements $\alpha(g,h)$, $g,h\in G$ are regular elements. Denote $S=\operatorname{reg}A_0$. Since A_0 is semiprime Goldie, $S^{-1}A_0$ is semisimple Artinian. From Theorem 9, $S^{-1}A$ is semisimple Artinian, in particular, it is Noetherian. Let I be an ideal in A and consider $(S^{-1}A)I$. Claim: this is an ideal. Let $s\in S$ and consider the following chain:

$$\left(S^{-1}A\right)I\subset\left(S^{-1}A\right)Is^{-1}\subset\left(S^{-1}A\right)Is^{-2}\subset\cdots.$$

This implies that $(S^{-1}A)Is^{-n}=(S^{-1}A)Is^{-m}$, $m>n$, and so $(S^{-1}A)I=(S^{-1}A)Is^{n-m}$. We get $(S^{-1}A)I(S^{-1}A)\subset(S^{-1}A)I$, or equivalently, $(S^{-1}A)I$ is an ideal in $S^{-1}A$. If J is the nilradical of A, then $(S^{-1}\cdot J)^n=S^{-1}\cdot J^n$ follows. For some n we have that $(S^{-1}\cdot J)^n=0$ in the semisimple Artinian ring $S^{-1}A$, thus $S^{-1}A\cdot J=0$ and $J=0$. □

Corollary 13. *If A is a CGR with A_0 a commutative Dedekind domain, G a finite group,* $\operatorname{char}A_0$ *does not divide $|G|$, then A is semiprime.*

Proposition 14. *Keep the conditions of Theorem* 12*. Then, $\forall P$, prime ideal of $S_{\mathrm{reg}}^{-1}A$: $P\cap A$ is a prime ideal of A.*

Proof. Define $S=\operatorname{reg}A_0$. Let P be a prime of $S^{-1}A$. Suppose that for ideals I and J of A, $IJ\subset P\cap A$. Then $S^{-1}A\cdot IJ\subset P$, hence $(S^{-1}A\cdot I)(S^{-1}A\cdot J)\subset P$. Therefore $S^{-1}A\cdot I\subset P$ if $S^{-1}A\cdot J\not\subset P$. This means $I\subset P\cap A$ if $J\not\subset P\cap A$ and conversely. □

Remark 15. The situation of Theorem 12 arises when A is centrally crystalline graded over the semiprime Goldie Ring A_0 and $\operatorname{char}A_0$ does not divide $|G|$, such that A (or A_0) is a P.I.-ring.

3. Spectrally twisted groups

In this section we will introduce an object closely related to a group. This object, the spectrally twisted group (STG) has more than one operation on it, and those operations are not necessarily associative, but they are connected.

3.1. Some definitions, conventions and properties

Definition 16 *(Spectrally twisted group).* Consider $\{\Gamma,\theta,G,F,\{M_P\}_{P\in F}\}$. Γ and F are sets. $G,\cdot$ is a group, acting on the right on F. θ is a surjective map from Γ to G. $\{M_P\}_{P\in F}$ are maps from $\Gamma\times\Gamma$ to Γ. They satisfy $\forall P\in F$, $\forall x,y,z\in\Gamma$:

$$M_P\left[M_P(x,y),z\right]=M_P\left[x,M_{P\theta(x)}(y,z)\right],$$
$$\theta\left[M_P(x,y)\right]=\theta(x)\cdot\theta(y).$$

For ease of notation, we will write $\forall P\in F$, $\forall x,y\in\Gamma$:

$$\theta_x:=\theta(x),$$
$$(x,y)_P:=M_P(x,y).$$

$\exists e_\Gamma\in\Gamma$: $\forall P\in F$, $\forall x\in\Gamma$: $(x,e_\Gamma)_P=x=(e_\Gamma,x)_P$.

Given a $P\in F$, $x\in\Gamma$, $\exists\,{}^Px, x^P\in\Gamma$, the left resp. right inverse of x, which satisfy

$$\left({}^Px,x\right)_P=e_\Gamma=\left(x,x^P\right).$$

We call $\{\Gamma,\theta,G,F,\{M_P\}_{P\in F}\}$ a **spectrally twisted group**. If there is no confusion possible, we only write Γ to denote the STG.

Lemma 17. *With notations as above, we have* $\forall x \in \Gamma$, $\forall P \in F$:

1. e_Γ *is unique.*
2. *If* y *is a left or right inverse to* x *for* P, *then* $\theta_y = \theta_x^{-1}$.
3. ${}^P x = x^{P\theta_x^{-1}}$.
4. ${}^P x$ *and* x^P *are unique.*

Proof. 1. Suppose e'_Γ is also an identity element, then $\forall P \in F$:

$$e'_\Gamma = \left(e'_\Gamma, e_\Gamma\right)_P = e_\Gamma.$$

2. $(y,x)_P = e_\Gamma \Rightarrow \theta_y \cdot \theta_x = \theta_{e_\Gamma} = e_G \Rightarrow \theta_y = \theta_x^{-1}$.
3. Fix $x \in \Gamma$ and $P \in F$:

$$\begin{aligned} {}^P x = \left({}^P x, e_\Gamma\right)_P &= \left[{}^P x, \left(x, x^{P\theta_x^{-1}}\right)_{P\theta_x^{-1}}\right]_P \\ &= \left[\left({}^P x, x\right)_P, x^{P\theta_x^{-1}}\right]_P = x^{P\theta_x^{-1}}. \end{aligned}$$

4. Suppose $(y,x)_P = e_\Gamma$ then

$$\begin{aligned} y &= (y, e_\Gamma)_P \\ &= \left[y, \left({}^P x, x\right)_P\right]_P \\ &= \left[y, \left(x, x^{P\theta_x^{-1}}\right)_{P\theta_x^{-1}}\right]_P \\ &= \left[(y,x)_P, x^{P\theta_x^{-1}}\right]_P \\ &= x^{P\theta_x^{-1}} \\ &= {}^P x. \end{aligned}$$

Similarly for a right inverse. □

Remark 18. A group $\kappa, \cdot$ is also an STG if we set $G = \kappa$, $\theta = \mathrm{Id}_\kappa$, $F = P$, $M_P =$ " $\cdot$ ".

3.2. Morphisms

Definition 19. Consider two STG's

$$\begin{gathered} \left\{\Gamma, \theta, G, F, \{M_P\}_{P \in F}\right\}, \\ \left\{\Gamma', \theta', G', F', \left\{M'_P\right\}_{P \in F'}\right\}. \end{gathered}$$

Then $f = \{f_\Gamma, f_G, f_F\}$ is a **morphism of STG's** if:

1. $f_\Gamma : \Gamma \to \Gamma'$ satisfies

$$f_\Gamma\left(M_P(x,y)\right) = M'_{f_F(P)}\left(f_\Gamma(x), f_\Gamma(y)\right).$$

2. $f_G : G \to G'$ satisfies

$$f_G\big(\theta(x)\cdot\theta(y)\big) = \theta'\big(f_G(x)\big)\cdot\theta'\big(f_G(y)\big).$$

3. $f_F : F \to F'$ satisfies

$$f_F(P\theta_x) = f_F(P)\theta'_{f_G(x)}.$$

For ease of notation, we use f without subscript, even if we mean f_Γ, f_G, f_F.

Definition 20. Let f be a morphism, then

$$\operatorname{Ker} f := \big\{x\in\Gamma \bigm| f_\Gamma(x) = e_{\Gamma'}\big\}.$$

In the definition, θ is a map from Γ to its twisting group G. But θ completely destroys the spectral twist. We introduce a family of morphisms that retain a bit of that structure:

Definition 21. Let Γ be an STG, and let $T,\cdot$ be a group. Consider the family $\psi := \{\psi_P\}_{P\in F}$ of maps from Γ to T. Suppose that $\forall P\in F$, $\forall x, y\in\Gamma$ we have

$$\psi_P\big((x,y)_P\big) = \psi_P(x)\cdot\psi_{P\theta_x}(y),$$

then we call the family $\{\psi_P\}_{P\in F}$ a **spectrally twisted multiplicative map** (STMM).

Remark 22. This definition does not conflict with the spectral twist of the $\{M_P\}_{P\in F}$, since for $P\in F$, $x, y, z\in\Gamma$:

$$\begin{aligned}\psi_P\big[\big((x,y)_P,z\big)_P\big] &= \psi_P\big[(x,y)_P\big]\psi_{P\theta_x\theta_y}(z)\\ &= \psi_P(x)\psi_{P\theta_x}(y)\psi_{P\theta_x\theta_y}(z)\\ &= \psi_P(x)\psi_{P\theta_x}\big[(y,z)_{P\theta_x}\big]\\ &= \psi_P\big[x,(y,z)_{P\theta_x}\big].\end{aligned}$$

Lemma 23. *With notations as above, we have* $\forall x\in\Gamma$, $\forall P\in F$:

1. $\psi_P(e_\Gamma) = e_T$.
2. $\psi_{P\theta_x}(x^P) = (\psi_P(x))^{-1}$.
3. $\psi_P({}^P x) = (\psi_{P\theta_x^{-1}}(x))^{-1}$.

Proof. Straightforward calculation. $\square$

Definition 24. Let Γ be an SGT, T a group. $\psi = \{\psi_P\}_{P\in F}$ an STMM from Γ to T, then we define

$$\operatorname{Ker}\psi_P := \big\{x\in\Gamma \bigm| \psi_P(x) = e_T\big\}.$$

3.3. Subgroups

Let $\{\Gamma,\theta,G,F,\{M_P\}_{P\in F}\}$ be an STG. Suppose $\chi\subset\Gamma$ is closed under $\{M_P\}_{P\in F}$, $e_\Gamma\in\chi$ and $\forall x\in\chi$, $\forall P\in F$: ${}^P x, x^P\in\chi$. It is easy to prove that $\theta(\chi)$ is a subgroup of G in this case.

Definition 25. Considering the above, we say $\{\chi,\theta_{|\chi},\theta(\chi),F,\{M_P\}_{P\in F}\}$ is a **subgroup** of $\{\Gamma,\theta,G,F,\{M_P\}_{P\in F}\}$. We write $\chi\leqslant\Gamma$.

4. Maximal graded orders

4.1. Spectrally twisted cocycles

Let D be a Dedekind domain, $\alpha : G \times G \Rightarrow D \backslash \{0\}$ be a generalized twisted 2-cocycle, i.e.

$$\alpha(g,h)\alpha(gh,t) = \sigma_g\big(\alpha(h,t)\big)\alpha(g,ht), \quad \forall g,h,t \in G.$$

Define

$$k : \operatorname{Spec} D \times G \times G \to \mathbb{Z} : (P,g,h) \mapsto k_P(g,h),$$

by the decomposition in prime ideals of $D\alpha(g,h)$, i.e.

$$I_{g,h} = D\alpha(g,h) = \prod_{P \in \operatorname{Spec} D} P^{k_P(g,h)}.$$

We find the following relation for $g,h,t \in G$, $P \in \operatorname{Spec} D$:

$$\begin{aligned}
&D\alpha(g,h)\alpha(gh,t) = D\sigma_g\big(\alpha(h,t)\big)\alpha(g,ht) \\
&\quad\Rightarrow \quad \prod_P P^{k_P(g,h)} \prod_P P^{k_P(gh,t)} = \prod_P \sigma_g(P)^{k_P(h,t)} \prod_P P^{k_P(g,ht)} \\
&\quad\overset{\text{unicity of decomposition}}{\Rightarrow} \quad k_P(g,h) + k_P(gh,t) = k_{P^g}(h,t) + k_P(g,ht).
\end{aligned}$$

Definition 26. Let R be a ring, G a group with action on $\operatorname{Spec} R$, then a map $k : \operatorname{Spec} R \times G \times G \to \mathbb{Z}$ is called a **spectrally twisted 2-cocycle** if and only if $\forall g,h,t \in G$ and $\forall P \in \operatorname{Spec} R$:

$$k_P(g,h) + k_P(gh,t) = k_{P^g}(h,t) + k_P(g,ht).$$

We need to find an equivalence relation between such spectrally twisted 2-cocycles. We base this on the equivalence relation for twisted 2-cocycles as follows. Let D be a Dedekind domain. Let $g,h \in G$, $\alpha, \beta : G \times G \to D$ be twisted generalized 2-cocycles, $\mu : G \to D$ a map and $P \in \operatorname{Spec} D$:

$$\begin{aligned}
&\beta(g,h)\mu(gh) = \alpha(g,h)\mu(g)\sigma_g\big(\mu(h)\big) \\
&\quad\Rightarrow \quad \prod_P P^{k'_P(g,h)} \prod_P P^{\lambda_P(gh)} = \prod_P P^{k_P(g,h)} \prod_P P^{\lambda_P(g)} \prod_P \sigma_g(P)^{\lambda_P(h)} \\
&\quad\Rightarrow \quad k'_P(g,h) + \lambda_P(gh) = k_P(g,h) + \lambda_P(g) + \lambda_{P^g}(h).
\end{aligned}$$

This last equation gives us a suitable equivalence relation.

Definition 27. For a group G and a Dedekind domain D, let k and k' be spectrally twisted 2-cocycles. Then k and k' are said to be **equivalent 2-cocycles** if and only if there exists a map $\lambda : \operatorname{Spec} D \times G \to \mathbb{Z}$ which satisfies $\forall g,h \in G$:

$$k'_P(g,h) + \lambda_P(gh) = k_P(g,h) + \lambda_P(g) + \lambda_{P^g}(h).$$

Note that since the map μ giving the equivalence on the level of the twisted 2-cocycles is not unique, the map λ is also not necessarily unique.

Now if $|G| = n$, define the set $\widetilde{G} := n^{-1}\mathbb{Z} \times G$ and $\forall P \in \operatorname{Spec} D$ operations M_P defined by

$$M_P : \widetilde{G} \times \widetilde{G} \to \widetilde{G} : \big[(a, g), (b, h)\big] \mapsto \big(a + b + k_P(g, h), gh\big).$$

Lemma 28. *The operation M_P defined as above is not associative but satisfies $\forall a, b, c \in Z$, $\forall g, h, t \in G$:*

$$M_P\big[M_P\big[(a, g), (b, h)\big], (c, t)\big] = M_P\big[(a, g), M_{Pg}\big[(b, h), (c, t)\big]\big]. \tag{1}$$

Proof.

$$\begin{aligned} M_P\big[M_P\big[(a, g), (b, h)\big], (c, t)\big] &= M_P\big[\big(a + b + k_P(g, h), gh\big), (c, t)\big] \\ &= \big(a + b + c + k_P(g, h) + k_P(gh, t), ght\big) \\ &= \big(a + b + c + k_P(g, ht) + k_{Pg}(h, t), ght\big) \\ &= M_P\big[(a, g), \big(b + c + k_{Pg}(h, t), ht\big)\big] \\ &= M_P\big[\big(a + b + k_P(g, h), gh\big), (c, t)\big]. \qquad \square \end{aligned}$$

We can define $\forall P \in \operatorname{Spec} D$:

$$i : n^{-1}\mathbb{Z} \to \widetilde{G} : a \mapsto (a, e),$$
$$\pi : \widetilde{G} \to G : (a, g) \mapsto g.$$

Lemma 29. *The element $(0, e)$ acts as a neutral element for M_P, $\forall P \in \operatorname{Spec} D$. Let $(a, g) \in \widetilde{G}$, then the left inverse is $(-a - k_{Pg}(g, g^{-1}), g^{-1})$, and the right inverse is $(-a - k_P(g, g^{-1}), g^{-1})$.*

Proof. Easy calculation. $\square$

Proposition 30. *$\{\widetilde{G}, \pi, G, \operatorname{Spec} D, \{M_P\}_{P \in \operatorname{Spec} D}\}$ is an STG.*

Proof. Obvious. $\square$

Now let $|G| = n$, then α^n is equivalent to the trivial 2-cocycle, and by taking powers, it is clear that nk_P as defined above is equivalent to the trivial 2-cocycle $\forall P \in \operatorname{Spec} D$. This means that $\forall P \in \operatorname{Spec} D$ we find a map $\gamma_P : G \to \mathbb{Z}$ with

$$nk_P(g, h) = \gamma_P(g) + \gamma_{Pg}(h) - \gamma_P(gh).$$

It is clear that $\gamma_P(e) = 0$, $\forall P \in \operatorname{Spec} D$. Note that this map γ not necessarily is unique! In the construction below, we fix such a γ.

Consider the direct product $n^{-1}\mathbb{Z} \times G$ and define $\forall P \in \operatorname{Spec} D$:

$$\phi_P : \widetilde{G} \to n^{-1}\mathbb{Z} \times G : (a, g) \mapsto \big(a + n^{-1}\gamma_P(g), g\big),$$
$$\phi_P\big[\big((a, g), (b, h)\big)_P\big] = \phi_P(a, g)\phi_{Pg}(b, h).$$

We see that $\{\phi_P \mid P \in \operatorname{Spec} D\}$ is a spectrally twisted multiplicative map. ϕ_P induces a spectrally twisted multiplicative map

$$\psi_P : \widetilde{G} \to n^{-1}\mathbb{Z} : (a, g) \mapsto a + n^{-1}\gamma_P(g).$$

It of course has the property $\psi_P \circ i = \mathrm{Id}$.

Fix $P \in \operatorname{Spec} D$. Now, given $g \in G$, choose $a_P(g) \in \mathbb{Z}$ such that

$$0 \leqslant \psi_P\big(a_P(g), g\big) < 1.$$

Now define $\forall P \in \operatorname{Spec} D$ the P-section of π

$$s_P : G \to \widetilde{G} : g \mapsto \big(a_P(g), g\big).$$

Since $\psi_P(a, e) = a + 0$, $s_P(e) = (0, e)$.

$$m_P : G \times G \to \mathbb{Z}\colon\ i\big(m_P(g, h)\big) = \big(\big(s_P(g), s_{Pg}(h)\big)_P, {}^P\big(s_P(gh)\big)\big)_P.$$

We calculate for $g, h \in G$, $P \in \operatorname{Spec} D$:

$$\begin{aligned}
&\big(\big(s_P(g), s_{Pg}(h)\big)_P, {}^P\big(s_P(gh)\big)\big)_P \\
&\quad = \big(\big(\big(a_P(g), g\big), \big(a_{Pg}(h), h\big)\big)_P, \big[-a_P(gh) - k_P\big(gh, (gh)^{-1}\big), (gh)^{-1}\big]\big)_P \\
&\quad = \big(\big[a_P(g) + a_{Pg}(h) + k_P(g, h), gh\big], \big[-a_P(gh) - k_P\big(gh, (gh)^{-1}\big), (gh)^{-1}\big]\big)_P \\
&\quad = \big[a_P(g) + a_{Pg}(h) - a_P(gh) + k_P(gh), e\big].
\end{aligned}$$

So if we define

$$m_P : G \times G \to \mathbb{Z}\colon\ i\big(m_P(g, h)\big) = \big(\big(s_P(g), s_{Pg}(h)\big)_P, {}^P\big(s_P(gh)\big)\big)_P,$$

then we find

$$m_P : G \times G \to \mathbb{Z}\colon\ i\big(m_P(g, h)\big) = a_P(g) + a_{Pg}(h) - a_P(gh) + k_P(g, h).$$

This means that $\forall P \in \operatorname{Spec} D$, m_P and k_P are equivalent 2-cocycles.

Definition 31. Let $P \in \operatorname{Spec} D$, then m_P as defined above is called the P**-maternal** 2**-cocycle** of $A = D \underset{\sigma,\alpha}{\diamond} G$ corresponding to γ. We call $a_P : G \to \mathbb{Z}$ the P**-maternal power** corresponding to γ.

For now, we fix γ.

Lemma 32. *Let $g \in G$, $a \in \mathbb{Z}$, $P \in \operatorname{Spec} D$, then*

$$\psi_{Pg}\big[(a, g)^{-1}\big] = -\psi_P\big[(a, g)\big].$$

Proof.

$$\begin{aligned}\psi_{Pg}\big[(a,g)^{-1}\big] &= \psi_{Pg}\big[-a-k_P\big(g,g^{-1}\big),g^{-1}\big]\\ &= -a-k_P\big(g,g^{-1}\big)+n^{-1}\gamma_{Pg}\big(g^{-1}\big)\\ &= -a-n^{-1}\gamma_P(g)\quad \big(nk_P\big(g,g^{-1}\big)=\gamma_P(g)+\gamma_{Pg}\big(g^{-1}\big)\big)\\ &= -\psi_P\big[(a,g)\big]. \qquad \square\end{aligned}$$

Proposition 33. *The spectrally twisted* 2*-cocycle* m_P *as defined above has its values in* $\{0,1\}$.

Proof.

$$\begin{aligned}\psi_P\big(i\big(m_P(g,h)\big)\big) &= \psi_P\big(\big(s_P(g)s_{Pg}(h)\big)s_P(gh)^{-1}\big)\\ &= \psi_P\big[s_P(g)s_{Pg}(h)\big]+\psi_{Pgh}\big[s_P(gh)^{-1}\big]\\ &\overset{\text{(Lemma 32)}}{=} \underbrace{\psi_P\big[s_P(g)\big]+\psi_{Pg}\big[s_{Pg}(h)\big]-\psi_P\big[s_P(gh)\big]}_{-1<\cdots<2}.\end{aligned}$$

Since by definition m_P has values in $\mathbb{Z}$, the proposition is proven. $\square$

4.2. Constructing the maximal graded orders

Let $A = D \underset{\sigma,\alpha}{\diamond} G$ be a crystalline graded ring with D a Dedekind domain. We will now give a few methods in constructing different maximal graded orders. The first method is by calculating for each possible γ the maternal order as defined in the previous section, then doing an additional calculation to find all graded orders, and as such, all maximal graded orders that contain this maternal order.

The second method uses a process of conjugation of an order with an element u_g as to obtain a new order. If we find a maximal graded order, all conjugates of this order are maximal, but as we shall see in the examples, not all contain the ring A.

4.2.1. Graded orders containing unital graded orders

Definition 34. Let $T = \bigoplus_{g\in G} I_g u_g$ be a graded order, $I_g = \prod_{P\in \operatorname{Spec} D} P^{r_P(g)}$. Then if, $\forall g,h\in G$ and $P\in \operatorname{Spec} D$:

$$r_P(g)+r_{Pg}(h)-r_P(gh)+k_P(g,h)\in\{0,1\},$$

we call T a **unital graded order**.

This section will present a calculation that, when given a unital order, we can construct a graded order above it or prove the order in question is maximal. We will do so by 'making zeroes' i.e. if $r_P(g)+r_{Pg}(h)-r_P(gh)+k_P(g,h)=1$ for some $g,h\in G$, $P\in \operatorname{Spec} D$, we will try to modify the r_P such that the new expression equals zero.

Consider $\forall P\in \operatorname{Spec} D$ the respective maternal 2-cocycles m_P and maternal powers a_P as defined above, corresponding to some fixed γ. Now construct $\forall g\in G$ (as above, all P are taken in $\operatorname{Spec} D$):

$$J_g = \prod_P P^{a_P(g)},$$

and if we take $\{u_g \mid g \in G\}$ the basis for A over D then define

$$M = \bigoplus_{g \in G} J_g u_g.$$

Definition 35. M defined as above is called the **maternal order** corresponding to γ. It is by construction unital.

This is a graded order. Suppose $T = \bigoplus_{g \in G} I_g u_g$ is a graded order. Then for each $g \in G$, I_g must be finitely generated, I_g is a fractional ideal. This is true for M since we only consider finitely many primes P (those that appear in the decomposition of the $\alpha(g,h)$) in the construction of m_P. (Not entirely true, but for the non-appearing primes we just take everything equal to 0.) This means that only finitely many $a_P(g)$ differ from 0. We need the following relation $\forall g,h \in G$

$$\begin{aligned} I_g u_g I_h u_h \subset I_{gh} u_{gh} \quad &\Leftrightarrow \quad I_g \sigma_g(I_h) u_g u_h \subset I_{gh} u_{gh} \\ &\Leftrightarrow \quad I_g \sigma_g(I_h) \alpha(g,h) u_{gh} \subset I_{gh} u_{gh} \\ &\Leftrightarrow \quad I_{gh}^{-1} I_g \sigma_g I_h \alpha(g,h) D \subset D. \end{aligned}$$

Now construct $\forall P \in \operatorname{Spec} D,\ g,h \in G$:

$$t_P(g,h) = -r_P(gh) + r_P(g) + r_{Pg}(h) + k_P(g,h),$$

where we set

$$I_g = \prod_P P^{r_P(g)}.$$

For T to be an order, $t_P(g,h) \geqslant 0$, $\forall g,h \in G, \forall P \in \operatorname{Spec} D$. This is true for the maternal order A by definition.

Suppose $T = \bigoplus_{g \in G} I_g u_g$ is a graded order with $M \subset T$, $M \neq T$. So we can assume that $\forall P \in \operatorname{Spec} D$ and all $g \in G$: $r_P(g) \leqslant a_P(g)$, and at least one P_0 prime and $g_0 \in G$ this inequality is strict. For the remainder we drop the subscript $\cdot_0$ if no confusion can arise. Since

$$\begin{aligned} t_P\big(g,g^{-1}\big) &= r_P(g) + r_{Pg}\big(g^{-1}\big) + k_P\big(g,g^{-1}\big) \\ &= \underbrace{r_P(g) - a_P(g)}_{<0} + \underbrace{r_{Pg}\big(g^{-1}\big) - a_{Pg}\big(g^{-1}\big)}_{\leqslant 0} + m_P\big(g,g^{-1}\big). \end{aligned}$$

This implies, since $t_P(g,g^{-1}) \geqslant 0$:

$$\begin{cases} m_P\big(g,g^{-1}\big) = 1, \\ r_{Pg}\big(g^{-1}\big) = a_{Pg}\big(g^{-1}\big), \\ a_P(g) = r_P(g) + 1, \\ t_P\big(g,g^{-1}\big) = 0. \end{cases} \tag{2}$$

So, to create a graded order $T \supset M$ where $T \neq M$, we must be able to find P prime, $g \in G$ with $m_P(g, g^{-1}) = 1$, otherwise we cannot satisfy (2). So, suppose we have $m_P(g, g^{-1}) = 1$ for some g, some P, and we have $T \supset M$. Then

$$t_P\big(g, g^{-1}\big) = \underbrace{r_P(g) - a_P(g)}_{\leqslant 0} + \underbrace{r_{Pg}\big(g^{-1}\big) - a_{Pg}\big(g^{-1}\big)}_{\leqslant 0} + \underbrace{m_P\big(g, g^{-1}\big)}_{=1} \geqslant 0.$$

We can now take either $r_P(g) - a_P(g) < 0$ or $r_{Pg}(g^{-1}) - a_{Pg}(g^{-1}) < 0$. Without loss of generality (since $m_P(g, g^{-1}) = m_{Pg}(g^{-1}, g)$), we can suppose:

$$r_P(g) = a_P(g) - 1,$$
$$r_{Pg}\big(g^{-1}\big) = a_{Pg}\big(g^{-1}\big).$$

But, we need to remark that this assumption might cause problems since T might not be a graded order any more. We need to investigate whether or not our changes make any of the $t_Q(A, B)$ negative, in which case we are not allowed to do the proposed change. Let $Q \in \operatorname{Spec} D$ and $A, B \in G$:

$$\begin{aligned} t_Q(A, B) = {} & \big[-r_Q(AB) + a_Q(AB)\big] \\ & + \big[r_Q(A) - a_Q(A)\big] \\ & + \big[r_{QA}(B) - a_{QA}(B)\big] + m_Q(A, B). \end{aligned} \tag{3}$$

To construct a maximal graded order $T \supset M$, we need to construct two main sets, U and V. U will contain all the couples (P, g) for which we can set $r_P(g) = a_P(g) - 1$. V will contain the couples (P, g) for which we cannot set $r_P(g) = a_P(g) - 1$, or which we always have to set $r_P(g) = a_P(g)$.

Add the (P, g) that cannot be modified, i.e. for which $m_P(g, g^{-1}) = 0$ to V. Now pick a couple (P, g) with $m_P(g, g^{-1}) = 1$. We will investigate if we can put (P, g) in U. We do so by checking if Eq. (3) does not give conflicts (i.e. $t_Q(A, B) < 0$).

To do this, we will focus on a couple (P, g) and whether or not it can be put in U. During the investigation, it is possible that we are faced with new investigations for couples (Q, h). We construct a set $W_{(P,g)}$ for such couples. Note that we need to check all elements in $W_{(P,g)}$ in a similar investigation before we can add (P, g) to U.

To start the investigation, we create sets $U_{(P,g)}$ and $V_{(P,g)}$ that will be copies of U and V, and $W_{(P,g)}$ for couples that need further investigation. These sets will change throughout the calculation. We then restart the process for all other possible couples not in V. For now, pick (P, g) not in V and put it in $U_{(P,g)}$. Put (Pg, g^{-1}) in $V_{(P,g)}$.

We have to (constantly) check if $\forall Q \in \operatorname{Spec} D$, $\forall A, B \in G$ the equation in (3) is positive if we would carry out $U_{(P,g)}$ and $V_{(P,g)}$ assuming $W_{(P,g)} \subset U_{(P,g)}$. It is obvious that the terms $m_Q(A, B)$ and $[-r_Q(AB) + a_Q(AB)]$ (whether or not they change) cannot make this expression negative. Changing $r_P(g)$ might cause a problem when

$$\begin{aligned} r_{QA}(B) = r_P(g) \quad & \Rightarrow \quad A = y \in G, \qquad B = g, \qquad Q = Ph^{-1}, \\ r_Q(A) = r_P(g) \quad & \Rightarrow \quad A = g, \qquad B = h \in G, \qquad Q = P. \end{aligned}$$

<u>$A = y \in G$, $B = g$, $Q = Ph^{-1}$.</u> We rewrite (3):

$$\begin{aligned} t_{Ph^{-1}}(h, g) = {} & \big[-r_{Ph^{-1}}(hg) + a_{Ph^{-1}}(hg)\big] \\ & + \big[r_{Ph^{-1}}(h) - a_{Ph^{-1}}(h)\big] \\ & + \big[r_P(g) - a_P(g)\big] + m_{Ph^{-1}}(h, g). \end{aligned}$$

There are two cases:

1. $m_{Ph^{-1}}(h,g)=1$.
2. $m_{Ph^{-1}}(h,g)=0$.

$\underline{m_{Ph^{-1}}(h,g)=1.}$ If $r_{Ph^{-1}}(h)-a_{Ph^{-1}}(h)=-1$ ((Ph^{-1},h) is in $U_{(P,g)}$) then we need to add (Ph^{-1},hg) to $U_{(P,g)}$. This needs a further check, so put (Ph^{-1},hg) in $W_{(P,g)}$.

$\underline{m_{Ph^{-1}}(h,g)=0.}$ We need to add (Ph^{-1},h) to $V_{(P,g)}$. If (Ph^{-1},h) is in $U_{(P,g)}$, put (P,g) in V. We can restart the investigation with a new (P,g).

We now want to put (Ph^{-1},hg) in $U_{(P,g)}$, and this can only be done if $m_{Ph^{-1}}(hg,(hg)^{-1})=1$. It will turn out this is always true. Our assumption leads to:

$$\underbrace{m_{Ph^{-1}}(h,g)}_{=0}+m_{Ph^{-1}}\big(hg,g^{-1}\big)=\underbrace{m_P\big(g,g^{-1}\big)}_{=1}+\underbrace{m_{Ph^{-1}}(h,e)}_{=0}.$$

And so we find

$$m_{Ph^{-1}}\big(hg,g^{-1}\big)=1. \tag{4}$$

We now consider 2 possibilities:

$$m_{Ph^{-1}}\big(h,h^{-1}\big)=m_P\big(h^{-1},h\big)=1,$$
$$m_{Ph^{-1}}\big(h,h^{-1}\big)=m_P\big(h^{-1},h\big)=0.$$

$\underline{m_{Ph^{-1}}(h,h^{-1})=m_P(h^{-1},h)=1.}$

$$\begin{cases} m_P\big(g,g^{-1}\big)=m_{Pg}\big(g^{-1},g\big)=1,\\ m_{Ph^{-1}}(h,g)=0,\\ m_{Ph^{-1}}\big(h,h^{-1}\big)=m_{Ph^{-1}}\big(h^{-1},h\big)=1.\end{cases}$$

From this we find

$$\underbrace{m_{Ph^{-1}}\big(hg,g^{-1}\big)}_{(4)\Rightarrow=1}+\underbrace{m_{Ph^{-1}}\big(h,h^{-1}\big)}_{=1}=m_{Pg}\big(g^{-1},h^{-1}\big)+m_{h^{-1}}\big(hg,(hg)^{-1}\big).$$

This means

$$m_{Ph^{-1}}\big(hg,(hg)^{-1}\big)=1.$$

We need to repeat the investigation with (Ph^{-1},hg) before we can put it in $U_{(P,g)}$. Put (Ph^{-1},hg) in $W_{(P,g)}$.

$\underline{m_{Ph^{-1}}(h,h^{-1})=m_P(h^{-1},h)=0.}$

$$\begin{cases} m_P\big(g,g^{-1}\big)=m_{Pg}\big(g^{-1},g\big)=1,\\ m_{h^{-1}P}(h,g)=0,\\ m_{h^{-1}P}\big(h,h^{-1}\big)=m_{h^{-1}P}\big(h^{-1},h\big)=0.\end{cases}$$

From this we find

$$m_{Pg}\big((hg)^{-1},h\big)+m_{Pg}\big(g^{-1},h^{-1}\big)=\underbrace{m_{Ph^{-1}}\big(h,h^{-1}\big)}_{=0},$$

and so

$$m_{Pg}\big(g^{-1},h^{-1}\big)=0.$$

This in time leads to

$$\underbrace{m_{Ph^{-1}}\big(hg,g^{-1}\big)}_{=1}+\underbrace{m_{Ph^{-1}}\big(h,h^{-1}\big)}_{=0}=\underbrace{m_{Pg}\big(g^{-1},h^{-1}\big)}_{=0}+m_{Ph^{-1}}\big(hg,(hg)^{-1}\big),$$

to find

$$m_{Ph^{-1}}\big(hg,(hg)^{-1}\big)=1.$$

We need to repeat the investigation with (Ph^{-1},hg) before we can put it in $U_{(P,g)}$. Put (Ph^{-1},hg) in $W_{(P,g)}$.

$\underline{A=g,\ B=h\in G,\ Q=P.}$ We rewrite (3)

$$\begin{aligned}t_P(g,h)=&\big[-r_P(gh)+a_P(gh)\big]\\&+\big[r_P(g)-a_P(g)\big]\\&+\big[r_{Pg}(h)-a_{Pg}(h)\big]+m_P(g,h).\end{aligned}$$

There are two cases:

1. $m_P(g,h)=1$.
2. $m_P(g,h)=0$.

$\underline{m_P(g,h)=1.}$ If $r_{Pg}(h)-a_{Pg}(h)=-1$ ((Pg,h) is in $U_{(P,g)}$) then we need to add (P,gh) to $U_{(P,g)}$. This needs a further check, so put (P,gh) in $W_{(P,g)}$.

$\underline{m_P(g,h)=0.}$ We need to add (Pg,h) to $V_{(P,g)}$. If (Pg,h) is in $U_{(P,g)}$, put (P,g) in V. We can restart the investigation with a new (P,g). We now want to put (P,gh) in $U_{(P,g)}$, and this can only be done if $m_P(gh,(gh)^{-1})=1$. It will turn out this is always true. Our assumption leads to:

$$\underbrace{m_P(g,h)}_{=0}+m_P\big(gh,(gh)^{-1}\big)=m_{Pg}\big(h,(gh)^{-1}\big)+\underbrace{m_P\big(g,g^{-1}\big)}_{=1}.$$

We need to repeat the investigation with (Ph^{-1},hg) before we can put it in $U_{(P,g)}$. Put (Ph^{-1},hg) in $W_{(P,g)}$.

We can repeat this process until we have found for all (P,g) not in V sets $U_{(P,g)}$, $V_{(P,g)}$ and $W_{(P,g)}$. The set

$$\kappa=\big\{(P,g)\;\big|\;P\in\operatorname{Spec}D,\ g\in G\big\}$$

is finite. Given a (P, g), we can look at all $V_{(Q,h)}$ that appear if we write out this tree:

$$\begin{array}{cccccccc}
 & & (P,g) & & & & & \\
 & \swarrow & \downarrow & \searrow & & & & \\
U_{(P,g)} & & V_{(P,g)} & & W_{(P,g)} & & & \\
 & & & \swarrow & \downarrow & \searrow & & \\
 & & (Q_1,h_1) & & \ldots & & (Q_s,h_s) & \\
 & \swarrow & \downarrow & \searrow & & \swarrow & \downarrow & \searrow
\end{array}$$

Then consider V' the union of all the $V_{(Q,h)}$ that appear in the full tree, and U' the union of all $U_{(Q,h)}$ and $W_{(Q,h)}$ that appear in the full tree. If $U' \cap V' \neq \emptyset$, there is a conflict somewhere and we put (P, g) in V, and investigate other (P, g). If $U' \cap V' = \emptyset$ then there are no conflicts whatsoever, and then we add all appearing $W_{(Q,h)}$ in the full tree to $U_{(P,g)}$, and all appearing $V_{(Q,h)}$ in the full tree to $V_{(P,h)}$. At last, we can add $U_{(P,g)}$ to U and $V_{(P,g)}$ to V. We can repeat the process with a new $(P, g) \in \kappa \backslash (U \cup V)$, until $U \cup V = \kappa$. If we look at the set $\{U_{(P,g)} \mid (P, g) \in U\}$, it contains a partition of U by construction. Every element of this partition $U_{(P,g)}$, represented by (P, g) determines one step in the reduction of M, corresponding to a chain of graded orders. If we do all changes suggested by elements of U, then we have found the top of our chain, a maximal graded order.

An algorithm. From these calculations we derive an algorithm for determining the unital maximal graded orders, given a crystalline graded ring $A = D \underset{\sigma,\alpha}{\diamond} G$, where D is a Dedekind domain.

1. Determine the relevant primes, namely those appearing in the prime decomposition of $\alpha(g, h)$, $g, h \in G$.
2. Determine for all the relevant primes P, the corresponding spectrally twisted 2-cocycle k_P from
$$D\big(\alpha(g,h)\big) = \prod_{P \in \operatorname{Spec} D} P^{k_P(g,h)}, \quad g,h \in G.$$
3. Calculate μ from (if $n = |G|$):
$$\alpha^n(g,h) = \mu(gh)^{-1}\mu(g)\sigma_g\big(\mu(h)\big).$$
4. Calculate for the relevant primes P from μ the γ_P from
$$D\big(\mu(g)\big) = \prod_{P \in \operatorname{Spec} D} P^{\gamma_P(g)}, \quad g \in G.$$
5. Calculate for the relevant primes P from γ_P the $a_P \in \mathbb{Z}$ from
$$0 \leqslant a_P(g) + \frac{1}{n}\gamma_P(g) < 1, \quad g \in G.$$
6. Calculate for the relevant primes P the spectrally twisted 2-cocycles m_P from
$$m_P(g,h) = a_P(g) + a_{Pg}(h) - a_P(gh) + k_P(g,h).$$
7. Determine for which relevant primes P and for which values $g \in G$ we have $m_P(g, g^{-1}) = 1$.

8. Construct U, the set of elements of

$$\kappa = \left\{(P, g) \;\middle|\; P \in \operatorname{Spec} D,\ g \in G\right\}$$

which we will modify ($r_P(g) = a_P(g) - 1$) and V the set of elements which we will not modify ($r_P(g) = a_P(g)$).
9. Set $U = \emptyset$, $V = \{(P, g) \mid m_P(g, g^{-1}) = 0\}$.
10. Pick $(P, g) \notin V$.
11. Construct $U_{(P,g)} = \{(P, g)\}$, $V_{(P,g)} = V \cup \{(Pg, g^{-1})\}$, $W_{(P,g)} = \emptyset$.
12. Find all $h \in G$ for which

$$m_{Ph^{-1}}(h, g) = 1.$$

If (Ph^{-1}, h) is in $U_{(P,g)}$ then add (Ph^{-1}, hg) to $W_{(P,g)}$. If $(Ph^{-1}, hg) \in V_{(P,g)}$, put (P, g) in V. Restart from step 10.
13. Find all $h \in G$ for which

$$m_{Ph^{-1}}(h, g) = 0.$$

Add (Ph^{-1}, h) to $V_{(P,g)}$. If (Ph^{-1}, h) is in $U_{(P,g)}$, put (P, g) in V. Restart from step 10. Else, add (Ph^{-1}, hg) to $W_{(P,g)}$.
14. Find all $h \in G$ for which

$$m_P(g, h) = 1.$$

If (Pg, h) is in $U_{(P,g)}$ then add (P, gh) to $W_{(P,g)}$. If $(P, gh) \in V_{(P,g)}$, put (P, g) in V. Restart from step 10.
15. Find all $h \in G$ for which

$$m_P(g, h) = 0.$$

Add (Pg, h) to $V_{(P,g)}$. If (Pg, h) is in $U_{(P,g)}$, put (P, g) in V. Restart from step 10. Else, add (P, gh) to $W_{(P,g)}$.
16. Repeat step 10 for all $(Q, h) \in W_{(P,g)}$. Since κ is finite, this loop ends in finite time.
17. We now have calculated $U_{(Q,h)}, V_{(Q,h)}, W_{(Q,h)}$ for all (Q, h) appearing in the full tree as described above. Now check for these (Q, h):

$$\left(\bigcup U_{(Q,h)} \cup \bigcup W_{(Q,h)}\right) \cap \bigcup V_{(Q,h)}.$$

If this intersection is empty, add all appearing $W_{(Q,h)}$ to $U_{(P,h)}$ and $U_{(P,g)}$ to U. Then add all appearing $V_{(Q,h)}$ to V. Restart from step 10. If this intersection is nonempty, add (P, g) to V and restart from step 10.
18. For all $(P, g) \in U$ set $r_P(g) = a_P(g) - 1$. For all $(P, g) \in V$ set $r_P(g) = a_P(g)$.
19. The graded order

$$T = \bigoplus_{g \in G} \left(\prod_{P \in \operatorname{Spec} D} P^{r_P(g)} \right) u_g$$

is a maximal graded order containing

$$M = \bigoplus_{g \in G} \left(\prod_{P \in \operatorname{Spec} D} P^{a_P(g)} \right) u_g.$$

Remark 36. It is clear that choosing (P, g) in step 10 determines the order T in some way. To find all maximal graded orders over M we need to use this algorithm for all possible choices (P, g) we make in various points of the algorithm.

Theorem 37. *Let $A = D \underset{\sigma,\alpha}{\diamond} G$ be a crystalline graded ring, where D is a commutative Dedekind domain, G a finite group. Then*:

1. *A unital order containing A is maternal.*
2. *All maximal graded orders containing A are unital.*

Proof.

1. Suppose that $T = \bigoplus_{g\in G} I_g u_g$ is a graded order, $I_g = \prod_{P\in \operatorname{Spec} D} P^{r_P(g)}$, $A \subset T$. We need to construct a function $\gamma : \operatorname{Spec} D \times G \to \mathbb{N} : (P, x) \mapsto \gamma_P(x)$ such that $\forall P \in \operatorname{Spec} D, \forall x, y \in G$ ($k_P(x, y)$ defined as usual):

$$n \cdot k_P(x, y) = \gamma_P(x) + \gamma_{Px}(y) - \gamma_P(xy),$$

and

$$0 \leqslant \frac{1}{n}\gamma_P(x) + r_P(x) < 1.$$

Since T is unital, $\forall P \in \operatorname{Spec} D, \forall x, y \in G$:

$$t_P(x, y) = k_P(x, y) + r_P(x) + r_{Px}(y) - r_P(xy) \in \{0, 1\}.$$

Define $\forall P \in \operatorname{Spec} D, \forall x \in G$, $\gamma_P(x)$ by

$$\gamma_P(x) = \left(\sum_{z\in G} t_P(x, z)\right) - nr_P(x) \geqslant 0.$$

Then $\forall P \in \operatorname{Spec} D, \forall x, y \in G$:

$$\begin{aligned}
&\gamma_P(x) + \gamma_{Px}(y) - \gamma_P(xy)\\
&\quad = \left(\sum_{z\in G} t_P(x, z) + \sum_{z\in G} t_{Px}(y, z) - \sum_{z\in G} t_P(xy, z)\right) - n\big(r_P(x) - r_{Px}(y) + r_P(xy)\big)\\
&\quad = \sum_{z\in G}\big(t_P(x, yz) + t_{Px}(y, z) - t_P(xy, z)\big) - nt_P(x, y) + nk_P(x, y)\\
&\quad = \left(\sum_{z\in G} t_P(x, y)\right) - nt_P(x, y) + nk_P(x, y) = nk_P(x, y).
\end{aligned}$$

Furthermore $\forall P \in \operatorname{Spec} D, \forall x \in G$:

$$0 \leqslant \frac{1}{n}\gamma_P(x) + r_P(x) = \frac{1}{n}\sum_{z\in G} t_P(x, z) < 1,$$

since $\forall P \in \operatorname{Spec} D, \forall x, y \in G$, $t_P(x, y) \in \{0, 1\}$ and $t_P(x, e) = 0$. In other words, T is a maternal order corresponding to γ.

2. Suppose that $T \supset A$ is a maximal graded order. Using the same notation as in 1 we construct $\forall P \in \operatorname{Spec} D$, $\forall x \in G$, $\gamma_P(x)$ by

$$\gamma_P(x) = \left(\sum_{z\in G} t_P(x,z)\right) - nr_P(x) \geqslant 0.$$

Construct the maternal order M corresponding to γ. $M = \bigoplus_{g\in G} J_g u_g$, $J_g = \prod_{P\in \operatorname{Spec} D} P^{a_P(g)}$. Fix $P \in \operatorname{Spec} D$ and $x \in G$. Suppose

$$0 \leqslant \frac{1}{n}\gamma_P(x) + r_P(x) < 1,$$

then $r_P(x) = a_P(x)$, since an integer with this property is unique. Suppose

$$\frac{1}{n}\gamma_P(x) + r_P(x) \geqslant 1 > \frac{1}{n}\gamma_P(x) + r_P(x),$$

then $r_P(x) > a_P(x)$. Combining these two, $\forall P \in \operatorname{Spec} D$, $\forall x \in G$: $0 \geqslant r_P(x) \geqslant a_P(x)$, implying $M \supset T$. Since T is maximal, $M = T$. $\square$

4.3. Conjugation method

In this section we will construct new orders by conjugating with u_g for some $g \in G$. We will therefore define a map Ψ_g for $g \in G$. Consider $A = D \underset{\sigma,\alpha}{\diamond} G$ be a crystalline graded ring with D a Dedekind domain, K the field of quotients of D. Let $g \in G$, then we have the following map on $K \underset{\sigma,\alpha}{\diamond} G$:

$$\Psi_g : K \underset{\sigma,\alpha}{\diamond} G \to K \underset{\sigma,\alpha}{\diamond} G : \sum_{x\in G} a_x u_x \mapsto u_g \sum_{x\in G} a_x u_x u_g^{-1}.$$

Proposition 38. *Let $g \in G$. Set* (*as in Proposition* 7):

$$H = \left\{g \in G \;\middle|\; \alpha\left(g, g^{-1}\right) \in U(D)\right\}.$$

Then, with notation as above:

1. *Ψ_g is an automorphism on $K \underset{\sigma,\alpha}{\diamond} G$ and is D^g-linear.* ($\{D^g = \{d \in D \mid \sigma_g(d) = d\}\}$.)
2. *$\Psi_h \in \operatorname{Aut}(A)$, $\forall h \in H$.*
3. *If $H \lhd G$, then Ψ_g is an automorphism on $S = D \underset{\sigma,\alpha}{\diamond} H$.*

Proof. 1. Very easy to verify.

2. Since $u_h \in U(A)$, this is clear.

3. An easy calculation yields ($g \in G$, $h \in H$):

$$u_g u_h u_g^{-1} = \alpha(g,h)\alpha^{-1}\left(ghg^{-1}, g\right)u_{ghg^{-1}}.$$

Since $H \lhd G$, $ghg^{-1} \in H$. From Proposition 7: $\alpha^{-1}(ghg^{-1}, g) \in D$ and as such $u_g u_h u_g^{-1} \in S$. $\square$

Let $A = D \underset{\sigma,\alpha}{\diamond} G$ be a crystalline graded ring with D a Dedekind domain. Consider a graded order T where $T = \bigoplus_{x\in G} I_x u_x$, $I_x = \prod_P P^{r_P(x)}$. We now conjugate with an element u_g for some $g \in G$.

In the following proposition, we will use these calculations for $x, y, g \in G$, $P \in \operatorname{Spec} D$:

Formula 1

$$\begin{aligned} \alpha\big(xg, x^{-1}\big)\alpha\big(xgx^{-1}, x\big) &= \sigma_{xg}\big(\alpha\big(x^{-1}, x\big)\big) \\ &= \sigma_{xgx^{-1}}\big(\sigma_x\big(\alpha\big(x^{-1}, x\big)\big)\big) \\ &= \sigma_{xgx^{-1}}\big(\alpha\big(x, x^{-1}\big)\big) \\ \Rightarrow \quad \alpha\big(xg, x^{-1}\big)\sigma_{xgx^{-1}}\big(\alpha^{-1}\big(x, x^{-1}\big)\big) &= \alpha^{-1}\big(xgx^{-1}, x\big). \end{aligned}$$

Formula 2

$$\begin{aligned} & t_P\big(g, g^{-1}xg\big) = r_P(g) + r_{Pg}\big(g^{-1}xg\big) - r_P(xg) + k_P\big(g, g^{-1}xg\big) \\ \Rightarrow \quad & r_{Pg}\big(g^{-1}xg\big) + k_P\big(g, g^{-1}xg\big) = t_P\big(g, g^{-1}xg\big) - r_P(g) + r_P(xg). \end{aligned}$$

Formula 3

$$\begin{aligned} & k_P(x, y) + k_P(xy, g) = k_{Px}(y, g) + k_P(x, yg) \\ \Rightarrow \quad & k_P(x, y) + k_P(xy, g) - k_{Px}(y, g) = k_P(x, yg). \end{aligned}$$

Formula 4

$$\begin{aligned} & t_P(x, g) = r_P(x) + r_{Px}(g) - r_P(xg) + k_P(x, g) \\ \Rightarrow \quad & -t_P(x, g) + r_P(x) = -r_{Px}(g) + r_P(xg) - k_P(x, g). \end{aligned}$$

Formula 5

$$t_P(x, yg) = r_P(x) + r_{Px}(yg) - r_P(xyg) + k_P(x, yg).$$

Formula 6

$$t_P(x, g) + t_P\big(xg, g^{-1}yg\big) = t_{Px}\big(g, g^{-1}yg\big) + t_P(x, yg).$$

Formula 7

$$t_P\big(g, g^{-1}xg\big) + t_P\big(xg, g^{-1}yg\big) = t_{Pg}\big(g^{-1}xg, g^{-1}yg\big) + t_P\big(g, g^{-1}xyg\big).$$

Proposition 39. *Define $\Psi_g(T)$ to be $u_g T u_g^{-1}$. Then $\Psi_g(T) = \bigoplus_{x \in G} \tilde{I}_x u_x$ is an order. If we set for $x, y \in G$ and $P \in \operatorname{Spec} D$:*

$$\begin{aligned} \tilde{I}_x &= \prod_P P^{\tilde{r}_P(x)}, \\ \tilde{t}_P(x, y) &= \tilde{r}_P(x) + \tilde{r}_{Px}(y) - \tilde{r}_P(xy) + k_P(x, y), \end{aligned}$$

we find

$$\begin{aligned}
\tilde{I}_x &= \sigma_g(I_{g^{-1}xg})\alpha\big(g, g^{-1}xg\big)\alpha^{-1}(x, g),\\
\tilde{r}_P(x) &= r_{Pg}\big(g^{-1}xg\big) + k_P\big(g, g^{-1}xg\big) - k_P(x, g),\\
\tilde{t}_P(x, y) &= t_{Pg}\big(g^{-1}xg, g^{-1}yg\big).
\end{aligned}$$

Proof. To make reading easier, we use brackets to do some grouping.

$$\begin{aligned}
u_g T u_g^{-1} &= \bigoplus_{x\in G} u_g I_x u_x u_g^{-1}\\
&= \bigoplus_{x\in G} \sigma_g(I_x) u_g u_x u_g^{-1}\\
&\overset{1}{=} \bigoplus_{x\in G} \sigma_g(I_x)\alpha(g, x)\alpha^{-1}\big(gxg^{-1}, g\big)u_{g^{-1}xg^{-1}}\\
\Rightarrow \quad & I_x = \sigma_g(I_{g^{-1}xg})\alpha\big(g, g^{-1}xg\big)\alpha^{-1}(x, g)\\
\Rightarrow \quad & \tilde{r}_P(x) = r_{Pg}\big(g^{-1}xg\big) + k_P\big(g, g^{-1}xg\big) - k_P(x, g),
\end{aligned}$$

$$\begin{aligned}
\tilde{t}_P(x, y) &= \tilde{r}_P(x) + \tilde{r}_{Px}(y) - \tilde{r}_P(xy) + k_P(x, y)\\
&= \big[r_{Pg}\big(g^{-1}xg\big) + k_P\big(g, g^{-1}xg\big)\big] - k_P(x, g)\\
&\quad + \big[r_{Pxg}\big(g^{-1}yg\big) + k_{Px}\big(g, g^{-1}yg\big)\big] - k_{Px}(y, g)\\
&\quad + \big[-r_{Pg}\big(g^{-1}xyg\big) - k_P\big(g, g^{-1}xyg\big)\big] + k_P(xy, g)\\
&\quad + k_P(x, y)\\
&\overset{2}{=} t_P\big(g, g^{-1}xg\big) - r_P(g) + r_P(xg) - k_P(x, g)\\
&\quad + t_{Px}\big(g, g^{-1}yg\big) - r_{Px}(g) + r_{Px}(yg)\\
&\quad - t_P\big(g, g^{-1}xyg\big) + r_P(g) - r_P(xyg)\\
&\quad + \big[-k_{Px}(y, g) + k_P(xy, g) + k_P(x, y)\big]\\
&\overset{3}{=} t_P\big(g, g^{-1}xg\big) + t_{Px}\big(g, g^{-1}yg\big) - t_P\big(g, g^{-1}xyg\big)\\
&\quad + \big[r_P(xg) - r_{Px}(g) - k_P(x, g)\big]\\
&\quad + \big[r_{Px}(yg) - r_P(xyg) + k_P(x, yg)\big]\\
&\overset{4,5}{=} t_P\big(g, g^{-1}xg\big) - t_P\big(g, g^{-1}xyg\big) + r_P(x) - r_P(x)\\
&\quad + \big[t_{Px}\big(g, g^{-1}yg\big) - t_P(x, g) + t_P(x, yg)\big]\\
&\overset{6}{=} t_P\big(xg, g^{-1}yg\big) + t_P\big(g, g^{-1}xg\big) - t_P\big(g, g^{-1}xyg\big)\\
&\overset{7}{=} t_{Pg}\big(g^{-1}xg, g^{-1}yg\big). \qquad \square
\end{aligned}$$

Let $A = D \diamondsuit_{\sigma,\alpha} G$ be a crystalline graded ring with D a Dedekind domain. It is obvious that a maximal graded order will stay maximal after conjugation. But it happens that the original ring A will not be contained in the conjugated order, see example 5.3.

4.4. A special case

For this section, we take $A = D \diamondsuit_{\sigma,\alpha} G$ be a crystalline graded ring with D a Dedekind domain, but with a special restriction. Suppose that all prime ideals appearing in the decomposition of the $\alpha(g,h)$, $g,h \in G$ are invariant for the group action, i.e. $Pg = P$, $\forall g \in G$, $\forall P$ appearing in the decomposition of the twisted 2-cocycle. This is the case if D is a Discrete Valuation Ring, or if all appearing primes P are of the form (a,b) where $a,b \in D^G$. In this case, all found additive 2-cocycles and corresponding prime powers as in the sections above are not spectrally twisted. α however, still is a twisted 2-cocycle.

Lemma 40. *With assumptions as above, suppose we have two additive 2-cocycles m, m' from $G \times G$ to $D \backslash \{0\}$ and suppose $\exists l_P : G \to \mathbb{Z}$ a map with*

$$m'_P(g,h) + l_P(gh) = m_P(g,h) + l_P(g) + l_{Pg}(h),$$

$\forall g,h \in G$. Furthermore, suppose $m'_P(e,e) = m_P(e,e) = 0$, $m_P(g,h) \in \{0,1\}$, $m'_P(g,h) \in \mathbb{N}$, $\forall P \in \operatorname{Spec} D$, $\forall g,h \in G$, then $l_P(g) \in \mathbb{N}$, $\forall P$, $\forall g,h \in G$.

Proof. Suppose $\exists P$, $\exists g \in G$ with $l_P(g) < 0$ then

$$l_P(g^2) + m'_P(g,g) = m_P(g,g) + l_P(g) + l_P(g) < 0,$$

since $Pg = P$ and $m_P(g,g) \in \{0,1\}$. We also have

$$l_P(g^3) + m'_P(g,g^2) = m_P(g,g^2) + l_P(g) + l_P(g^2).$$

Adding these expressions:

$$l_P(g^3) + m'_P(g,g^2) + m'_P(g,g) = m_P(g,g^2) + l_P(g) + l_P(g^2) + m'_P(g,g) < 0.$$

Continuing in this fashion we see ($|G| = n$):

$$l_P(g^n) + \sum_{i=1}^{n-1} m'_P(g,g^i) < 0,$$

and since $l_P(g^n) = l_P(e) = 0$ this is a contradiction with the fact that m'_P has values in $\mathbb{N}$. $\square$

We can construct from the given ring A the 2-cocycles k and m as before, but they will not be spectrally twisted. We construct a maternal order M given a γ.

Theorem 41. *With conventions as above, M is the unique maximal graded order over D containing A.*

Proof. Let T be a graded D-order in $K \diamondsuit_{\sigma,\alpha} G$, where K is the quotient field of D. We can associate to T the maps t and r like in Section 4.2.1, i.e.

$$T = \bigoplus J_g u_g, \quad J_g = \prod_{P \in \operatorname{Spec} D} P^{r_P(g)},$$

$$t_P(g,h) = -r_P(gh) + r_P(g) + r_P(h) + k_P(x,y) \geqslant 0.$$

This implies, since

$$m_P(g,h) = -a_P(gh) + a_P(g) + a_P(h) + k_P(g,h),$$

that

$$t_P(g,h) = -r_P(gh) + a_P(gh) + r_P(g) - a_P(g) + r_P(h) - a_P(h) + m_P(x,y) \geqslant 0,$$

$\forall P \in \operatorname{Spec} D$ and $\forall g,h \in G$. By the previous lemma we find $r_P(g) - a_P(g) \geqslant 0$, $\forall P \in \operatorname{Spec} D$, $\forall g \in G$. In other words, $\forall g \in G$:

$$J_g = \prod_{P \in \operatorname{Spec} D} P^{r_P(g)} \subset \prod_{P \in \operatorname{Spec} D} P^{a_P(g)} = I_g \quad \Rightarrow \quad T \subset A.$$

The theorem now follows. $\square$

5. Examples

5.1. Example 1: 1 maximal graded order

Consider $\mathbb{Z}[i] \underset{\sigma,\alpha}{\diamond} \mathbb{Z}_4$ where

$$\sigma : \mathbb{Z}_4 \to \operatorname{Aut}\mathbb{Z}[i] : \bar{0}, \bar{2} \mapsto \mathrm{Id}, \ , \bar{1}, \bar{3} \mapsto \bar{\cdot},$$

where $\bar{\cdot}$ is the standard conjugation in $\mathbb{C}$. We set α:

α	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	1	1	1	1
$\bar{1}$	1	2	$1-2i$	$2-4i$
$\bar{2}$	1	$1-2i$	5	$1+2i$
$\bar{3}$	1	$2+4i$	$1+2i$	2

The relevant primes are

$$P_1 = 1+2i,$$
$$P_2 = 1-2i,$$
$$P_3 = 1+i.$$

We calculate $k_{P_1}, k_{P_2}, k_{P_3}$:

k_{P_1}	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	0	0	0	0
$\bar{1}$	0	0	0	0
$\bar{2}$	0	0	1	1
$\bar{3}$	0	1	1	0

k_{P_2}	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	0	0	0	0
$\bar{1}$	0	0	1	1
$\bar{2}$	0	1	1	0
$\bar{3}$	0	0	0	0

k_{P_3}	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	0	0	0	0
$\bar{1}$	0	2	0	2
$\bar{2}$	0	0	0	0
$\bar{3}$	0	2	0	2

We calculate μ such that $\alpha^4(x, y) = \mu(xy)^{-1}\mu(x)\sigma_x(\mu(y))$. We have

$$\alpha^4(\bar{2},\bar{2}) = \mu(\bar{2})^2 = 625,$$
$$\alpha^4(\bar{1},\bar{3}) = \mu(\bar{1})\overline{\mu(\bar{3})} = 2^4(1-2i)^4,$$
$$\alpha^4(\bar{1},\bar{3}) = \overline{\mu(\bar{1})}\mu(\bar{3}) = 2^4(1+2i)^4,$$
$$\alpha^4(\bar{1},\bar{1}) = \mu(\bar{2})^{-1}\mu(\bar{1})\overline{\mu(\bar{1})} = \frac{1}{25}\left|\mu(\bar{1})\right|^2 = 16 \quad \Rightarrow \quad \left|\mu(\bar{1})\right|^2 = 400.$$

From which we can conclude

$$\mu(\bar{1}) = 4(1-2i)^2,$$
$$\mu(\bar{2}) = 25,$$
$$\mu(\bar{3}) = 4(1+2i)^2.$$

We now calculate γ and $1/4\gamma$:

γ	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
P_1	0	0	2	2
P_2	0	2	2	0
P_3	0	4	0	4

$\frac{1}{4}\gamma$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
P_1	0	0	$\frac{1}{2}$	$\frac{1}{2}$
P_2	0	$\frac{1}{2}$	$\frac{1}{2}$	0
P_3	0	1	0	1

And then a $(0 \leqslant a_{P_j} + 1/4\gamma_{P_j} < 1)$:

a	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
P_1	0	0	0	0
P_2	0	0	0	0
P_3	0	-1	0	-1

So the maternal order is

$$\mathbb{Z}[i]u_{\bar{0}} + (1+i)^{-1}\mathbb{Z}[i]u_{\bar{1}} + \mathbb{Z}[i]u_{\bar{2}} + (1+2)^{-1}\mathbb{Z}[i]u_{\bar{3}}.$$

We check for maximality:

$$m_{P_1}(\bar{3},\bar{1}) = 1,$$
$$m_{P_1}(\bar{2},\bar{2}) = 1,$$
$$m_{P_2}(\bar{1},\bar{3}) = 1,$$
$$m_{P_2}(\bar{2},\bar{2}) = 1,$$
$$t_{P_1}(\bar{2},\bar{2}) = r_{P_1}(\bar{2}) + r_{P_1}(\bar{2}) + k_{P_1}(\bar{2},\bar{2}),$$

and so $r_{P_1}(\bar{2}) = a_{P_1}(\bar{2})$ and similarly $r_{P_2}(\bar{2}) = a_{P_2}(\bar{2})$. We now check case (4) for $x = \bar{3}$, $P = P_1$.

$$
\begin{array}{llll}
m_{P_1}(\bar{3},\bar{1}) & x=\bar{3} & y=\bar{1} & m_{P_2}(\bar{1},\bar{3})=1\\
 & & y=\bar{2} & m_{P_1}(\bar{2},\bar{3})=1\\
 & & y=\bar{3} & m_{P_2}(\bar{3},\bar{3})=0
\end{array}
$$

This means we need to modify $r_{P_2}(\bar{3}+\bar{3})=r_{P_2}(\bar{2})$ which is impossible. The same problem appears in the case of $m_{P_2}(\bar{1},\bar{3})$. This means that the maternal order itself is maximal.

5.2. *Example 2: 2 maximal graded orders*

Consider $\mathbb{Z}[i] \underset{\sigma,\alpha}{\diamond} \mathbb{Z}_2$ where

$$\sigma:\mathbb{Z}_2 \to \operatorname{Aut}\mathbb{Z}[i]: \bar{0} \mapsto \mathrm{Id},\ ,\ \bar{1} \mapsto \bar{\cdot}.$$

Put $\alpha(\bar{1},\bar{1})=5$, then the relevant primes are $P_1=1+2i$, $P_2=1-2i$. Calculating the k yields $k_{P_1}(\bar{1},\bar{1})=k_{P_2}(\bar{1},\bar{1})=1$. $\alpha^2(\bar{1},\bar{1})=25$ and so $\mu(\bar{1})=5$. We now find that

$$\gamma_{P_1}(\bar{1})=1,\qquad \gamma_{P_2}(\bar{1})=1,\qquad a_{P_1}(\bar{1})=0,\qquad a_{P_2}(\bar{1})=0,\qquad m_{P_1}(\bar{1},\bar{1})=1,\qquad m_{P_2}(\bar{1},\bar{1})=1.$$

This means that the maternal order is

$$\mathbb{Z}[i]u_{\bar{0}}+\mathbb{Z}[i]u_{\bar{1}}.$$

We can either

1. $$r_{P_1}(\bar{1})=a_{P_1}(\bar{1})-1=-1,$$
   $$r_{P_2}(\bar{1})=a_{P_2}(\bar{1})=0,$$

2. $$r_{P_2}(\bar{1})=a_{P_2}(\bar{1})-1=-1,$$
   $$r_{P_1}(\bar{1})=a_{P_1}(\bar{1})=0.$$

And we get the following maximal graded orders

$$T_1=\mathbb{Z}[i]u_{\bar{0}}+(1+2i)^{-1}\mathbb{Z}[i]u_{\bar{1}},$$
$$T_2=\mathbb{Z}[i]u_{\bar{0}}+(1-2i)^{-1}\mathbb{Z}[i]u_{\bar{1}}.$$

As one can check, both are even strongly graded.

5.3. *Example 3: 2 maximal graded orders*

Consider $\mathbb{Z}[i] \underset{\sigma,\alpha}{\diamond} \mathbb{Z}_4$ where

$$\sigma:\mathbb{Z}_4 \to \operatorname{Aut}\mathbb{Z}[i]: \bar{0},\bar{2} \mapsto \mathrm{Id},\ ,\ \bar{1},\bar{3} \mapsto \bar{\cdot},$$

where $\bar{\cdot}$ is the standard complex conjugation. We set α:

α	0	1	2	3
0	1	1	1	1
1	1	4	$(1+i)(1+2i)$	$20(1+i)(1+2i)$
2	1	$(1+i)(1+2i)$	50	$5(1-i)(1-2i)$
3	1	$20(1-i)(1-2i)$	$5(1-i)(1-2i)$	20

The relevant primes are

$$P_1 = 1+2i,$$

$$P_2 = 1-2i,$$

$$P_3 = 1+i.$$

We calculate k_{P_1}, k_{P_2}, k_{P_3}:

k_{P_1}	0	1	2	3
0	0	0	0	0
1	0	0	1	2
2	0	1	2	1
3	0	1	1	1

k_{P_2}	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	2	2
3	0	2	2	1

k_{P_3}	0	1	2	3
0	0	0	0	0
1	0	4	1	5
2	0	1	2	1
3	0	5	1	4

From this we can calculate the possibilities for γ:

γ	0	1	2	3
P_1	0	s	4	s
P_2	0	$4-s$	4	$8-s$
P_3	0	10	4	10

where $s \in \mathbb{Z}$ and $0 \leqslant s \leqslant 4$. And so we can find 3 different maternal orders, namely one corresponding to $s=4$, call it M^A, one corresponding to $s=0$, M^B and one where $1 \leqslant s \leqslant 3$, L.

$$\begin{aligned}
M^A &= \mathbb{Z}[i]u_0 + (1+2i)^{-1}(1+i)^{-2}u_1 \\
&\quad + (1+2i)^{-1}(1-2i)^{-1}(1+i)^{-1}u_2 + (1+2i)^{-1}(1-2i)^{-1}(1+i)^{-2}u_3, \\
M^B &= \mathbb{Z}[i]u_0 + (1-2i)^{-1}(1+i)^{-2}u_1 \\
&\quad + (1-2i)^{-1}(1+2i)^{-1}(1+i)^{-1}u_2 + (1-2i)^{-2}(1+i)^{-2}u_3, \\
L &= \mathbb{Z}[i]u_0 + (1+i)^{-2}u_1 \\
&\quad + (1-2i)^{-1}(1+i)^{-1}(1+2i)^{-1}u_2 + (1-2i)^{-1}(1+i)^{-2}u_3.
\end{aligned}$$

Calculation and the method described in Section 4.2.1 reveals that M_1 and M_2 are maximal graded orders, and L is not. Depending on the choices made in the process of finding orders above L, we find M_1 and M_2.

Looking at conjugation, we find the following:

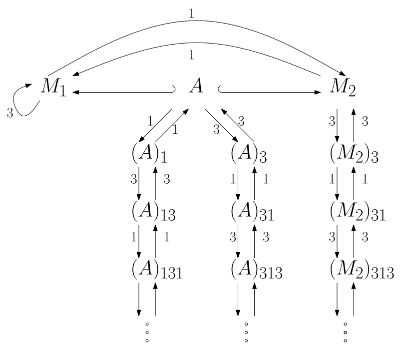
$$\Psi_3\big(M^A\big) = M^A,$$

$$\Psi_1\big(M^A\big) = M^B,$$

$$\Psi_1 \circ \Psi_1 = \mathrm{Id},$$

$$\Psi_3 \circ \Psi_3 = \mathrm{Id}.$$

We also find that $\Psi_3(\Psi_1(M^A))$ does not contain A anymore! Putting it in a picture:



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